which, combined with the three-points lemma (Lemma 9.11) with $\mathbf{a} = \mathbf{x}^{n+1}, \mathbf{b} = \mathbf{x}^n$, and $\mathbf{c} = \mathbf{x}$, yields the inequality

$$B_\omega(\mathbf{x}, \mathbf{x}^{n+1}) + B_\omega(\mathbf{x}^{n+1}, \mathbf{x}^n) - B_\omega(\mathbf{x}, \mathbf{x}^n) \leq \frac{m(\mathbf{x}, \mathbf{x}^n) - m(\mathbf{x}^{n+1}, \mathbf{x}^n) + g(\mathbf{x}) - g(\mathbf{x}^{n+1})}{L_n}.$$

Rearranging terms, we obtain that

$$\begin{aligned} m(\mathbf{x}^{n+1}, \mathbf{x}^n) + g(\mathbf{x}^{n+1}) + L_n B_\omega(\mathbf{x}^{n+1}, \mathbf{x}^n) \leq m(\mathbf{x}, \mathbf{x}^n) + g(\mathbf{x}) + L_n B_\omega(\mathbf{x}, \mathbf{x}^n) \\ - L_n B_\omega(\mathbf{x}, \mathbf{x}^{n+1}), \end{aligned}$$

which, combined with (10.94), yields the inequality

$$F(\mathbf{x}^{n+1}) \leq m(\mathbf{x}, \mathbf{x}^n) + g(\mathbf{x}) + L_n B_\omega(\mathbf{x}, \mathbf{x}^n) - L_n B_\omega(\mathbf{x}, \mathbf{x}^{n+1}).$$

Since f is convex, $m(\mathbf{x}, \mathbf{x}^n) \leq f(\mathbf{x})$, and hence

$$F(\mathbf{x}^{n+1}) - F(\mathbf{x}) \leq L_n B_\omega(\mathbf{x}, \mathbf{x}^n) - L_n B_\omega(\mathbf{x}, \mathbf{x}^{n+1}).$$

Plugging in $\mathbf{x} = \mathbf{x}^*$ and dividing by L_n, we obtain

$$\frac{F(\mathbf{x}^{n+1}) - F(\mathbf{x}^*)}{L_n} \leq B_\omega(\mathbf{x}^*, \mathbf{x}^n) - B_\omega(\mathbf{x}^*, \mathbf{x}^{n+1}).$$

Using the bound $L_n \leq \alpha L_f$ (see Remark 10.80),

$$\frac{F(\mathbf{x}^{n+1}) - F(\mathbf{x}^*)}{\alpha L_f} \leq B_\omega(\mathbf{x}^*, \mathbf{x}^n) - B_\omega(\mathbf{x}^*, \mathbf{x}^{n+1}),$$

and hence

$$F(\mathbf{x}^{n+1}) - F_{\text{opt}} \leq \alpha L_f B_\omega(\mathbf{x}^*, \mathbf{x}^n) - \alpha L_f B_\omega(\mathbf{x}^*, \mathbf{x}^{n+1}).$$

Summing the above inequality for $n = 0, 1, \ldots, k-1$, we obtain that

$$\sum_{n=0}^{k-1} (F(\mathbf{x}^{n+1}) - F_{\text{opt}}) \leq \alpha L_f B_\omega(\mathbf{x}^*, \mathbf{x}^0) - \alpha L_f B_\omega(\mathbf{x}^*, \mathbf{x}^k) \leq \alpha L_f B_\omega(\mathbf{x}^*, \mathbf{x}^0).$$

Using the monotonicity of the sequence of function values, we conclude that

$$k(F(\mathbf{x}^k) - F_{\text{opt}}) \leq \alpha L_f B_\omega(\mathbf{x}^*, \mathbf{x}^0),$$

thus obtaining the result

$$F(\mathbf{x}^k) - F_{\text{opt}} \leq \frac{\alpha L_f B_\omega(\mathbf{x}^*, \mathbf{x}^0)}{k}. \qquad \square$$